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# Synchronization of underdamped Josephson-junction arrays

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**Abstract.** Our recent experiments show that arrays of underdamped Josephson junctions radiate coherently only above a threshold number of junctions switched onto the radiating state. For each junction, the radiating state is a resonant step in the current-voltage characteristics due to the interaction between the junctions in the array and an electromagnetic cavity. Here we show that a model of a one-dimensional array of Josephson junctions coupled to a resonator can produce many features of the coherent behavior above threshold, including coherent radiation of power and the shape of the array current-voltage characteristic. The model also makes quantitative predictions about the degree of coherence of the junctions in the array. However, in this model there is no threshold; the experimental below-threshold region behavior could not be reproduced.

**PACS.** 74.50.+r Tunneling phenomena; point contacts, weak links, Josephson effects –
85.25.-j Superconducting devices

## 1 Introduction

A Josephson junction is a high-frequency generator: If a DC voltage $V_0$ is applied to it, it will generate an AC current with a frequency $f = (2e/h)V_0 = (483\ \text{GHz/mV})\ V_0$, where $h$ is Planck's constant [1]. Potential applications are for fast electronics, from high-speed digital devices to millimeter-wave circuits. In this paper we are concerned with high-frequency applications, in particular with the generation of millimeter and submillimeter waves. The two main problems for applications are that (i) the power generated by a single junction is too small, and (ii) that the linewidth of the emitted radiation is undesirably large.

One potential way to solve both the power and the linewidth problems is to construct arrays of many coherently oscillating junctions [2–9]. The power of $N$ coherently-emitting junctions can increase by $N^2$ over that of a single junction and the linewidth can also decrease (typically as $1/N$) for coherent $N$-junction systems. While it is relatively easy to fabricate many junctions on the same substrate, it is extremely difficult to have them oscillate coherently. The primary reason for this is that even with the same current bias, the different junctions will not have the same voltage (frequency) because of differences in the junction parameters.

To have an array oscillate coherently, it is necessary to exploit the non-linear properties of the Josephson junctions to provide coupling. One very attractive way to provide both strong coupling of the junctions as well as impedance matching to the high impedance environment of typical millimeter wave systems is to place the junctions along a planar transmission line, as shown in our previous experiments [8–10]. These arrays had two main features distinguishing them from typical two-dimensional arrays. First, the junctions had very low dissipation (orders of magnitudes smaller than the junctions in Refs. [5–7]). Second, each junction was strongly coupled to an electromagnetic mode in the transmission line formed by the array itself and the groundplane.

As discussed in detail in references [8–10], this coupling to a resonant mode produces hysteretic steps in the current voltage (I-V) characteristic. The steps are shown by the open circles in Figure 1a. The hysteresis in the junction I-V characteristics allows us to measure the emitted power as a function of the number of oscillating junctions, while junctions in conventional arrays cannot be controlled in this way. In the experiments the array is always biased on the resonant steps when the power coupled into the detector is measured as a function of the number of oscillating rows. The surprising result described in reference [8] is shown in Figure 1b. When a small number of oscillating junctions are biased on the resonant step, no detectable power is measured. Above a threshold number of junctions biased on the step, the array emits coherently and the power grows as the square of the number

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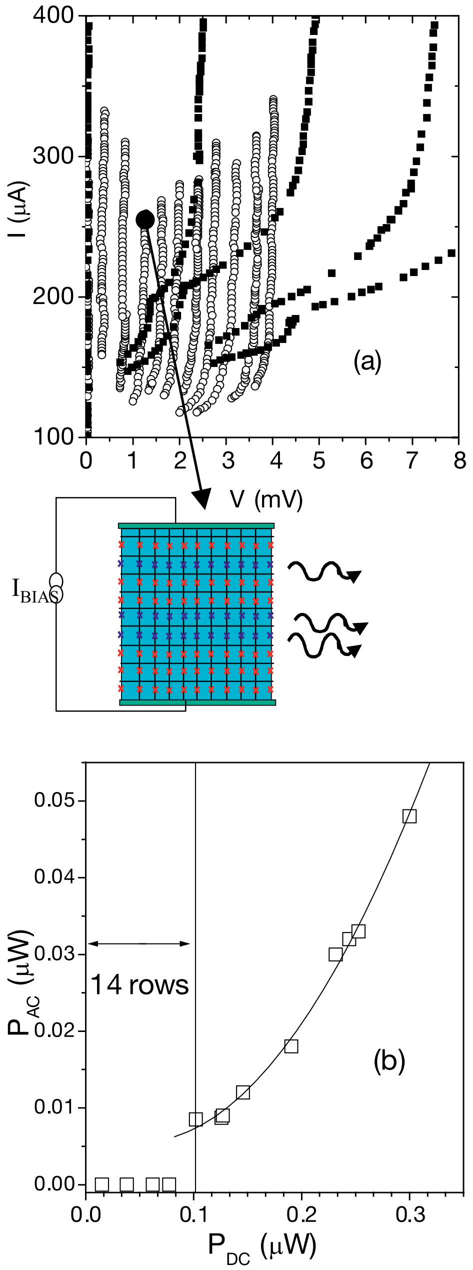


**Fig. 1.** (a) The low voltage region of a $10 \times 10$ array (dark squares). In the presence of an external magnetic field in the plane of the array, $H = 40$ Oe, sharp resonant steps appear (light circles). When biased on a resonant step, the junctions are oscillating *via* the Josephson effect. The sketch shows that when the array is biased on the third step, only three rows are oscillating. (b) AC power detected when biasing an increasing number of junctions on the resonant steps *vs.* the DC bias power. Data are from reference [8].

of active junctions, up to DC-to-AC conversion efficiency higher than 30% [9]. By contrast, high-dissipation arrays measured in references [5–7] show DC-to-AC conversion efficiency on the order of a few percent. Our later measurements on several other arrays [10] showed that by increasing the sensitivity of the detector it is possible to measure a small amount of incoherent power below threshold. In all our measurements, the junctions show different behavior in the emission of radiation below and above threshold: Above threshold, they emit coherently and the power increases dramatically [8–10].

In order to understand these experimental results, models of one-dimensional [11,12] and two-dimensional Josephson junction arrays [13] coupled to a resonant load have been studied. In all these models, the coherent regime yields results very similar to the experiments: I-V characteristics with resonant steps, as well as coherent radiation and a quantitative measure of the degree of coherence of the array, thus reproducing many of the experimental results and providing powerful new predictions. Filatrella *et al.* [13] also studied the comparison between the one-dimensional and the two-dimensional models and showed that there was no qualitative difference between the two, *i.e.* the one-dimensional model captured all the characteristics of the junction synchronization.

One outstanding issue is the explanation of the experimental threshold from the incoherent to the coherent state. Here we show that models in which junctions are globally coupled to a common cavity cannot reproduce the experimental threshold: Contrary to the experiments, in all these models the junctions are always synchronized when biased on the resonant step. This difference can only be discovered when carefully analyzing the calculated current-voltage characteristics of the models and comparing them to the experimental current-voltage characteristics. Here we will show this analysis in the case of a one-dimensional array coupled to a resonant load. The same qualitative picture holds for the two-dimensional model. This comparison and its implications are extremely important and they will be discussed below.

## 2 Simulations

Figure 2 shows a schematic of the one-dimensional model discussed in reference [11]. As shown previously, this model is essentially equivalent to the two-dimensional model. We use the one-dimensional model for computational simplicity. This circuit is a series array of Josephson oscillators with different natural frequencies (to account for fabrication imperfections) that are coupled *via* a simple resonant load. The load provides a feedback of a particular type: each junction interacts only *via* the mean field produced by the other oscillators. For simplicity we assume that all the junction resistances are the same, $R_j = R_0$ and similarly for the capacitances, $C_j = C_0$. The critical currents are randomly chosen from a uniform distribution with a width of $\pm 2\%$ about the average value. This yields the following set of equations for the circuit in Figure 2:

$$\beta_C \frac{\mathrm{d}^2\phi_j}{\mathrm{d}t^2} + \frac{\mathrm{d}\phi_j}{\mathrm{d}t} + I_{Cj} \sin\phi_j = \frac{I}{I_C} - \frac{\mathrm{d}q}{\mathrm{d}t} \quad j = 1, ...N_T, \tag{1a}$$

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{R}{L\omega_{RC}} \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC\omega_{RC}^2} q = \frac{1}{\beta_L} \sum_{i=1}^{N_T} \frac{\mathrm{d}\phi_i}{\mathrm{d}t} \cdot \tag{1b}$$

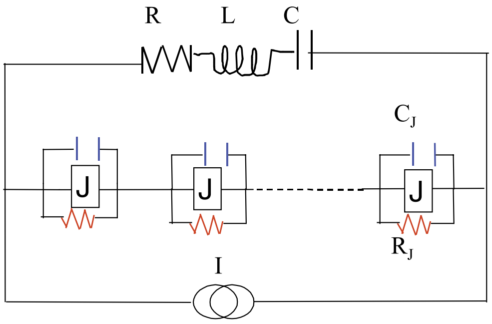


**Fig. 2.** Schematic of a one-dimensional array coupled to a lumped resonant circuit.

Here the $I_{C_j}$ are the critical currents of the junctions normalized to the average critical current $I_C$, $t$ is a dimensionless time, created by multiplying real time by the characteristic frequency $\omega_{RC} = 4\pi e R_0 I_C/h$, $q$ is a dimensionless charge, created by multiplying the real charge by $4\pi e R_0/h$, $\beta_C = 4\pi e I_C R_0^2 C_0/h$ is the usual Stuart-McCumber parameter, $\phi_j$ are the gauge invariant phase differences and $\beta_L = 4\pi e L I_C/h$ describes the coupling between the resonator and the Josephson junctions. $R$, $L$, and $C$ are the lumped elements forming the resonator.

Equation (1a) is simply the equation of motion for a shunted Josephson junction biased with a DC current $I$ shunted by an $RLC$ series circuit representing the cavity. Equation (1b) is the equation of motion for the series $RLC$ circuit subject to a voltage that is the sum of the voltages of the Josephson elements. The right-hand side of equation (1b) is the mean field generated by the Josephson circuit.

Equations (1) provide a compact description of a model system which is very useful because of its simplicity. We note that the two-dimensional version of this model yields qualitatively similar results [13]. We will use equations (1) to calculate $I-V$ curves, power output, and a measure of the degree of order in the arrays.

When biased at a fixed current, the junctions will have slightly different oscillation frequencies with average frequency $\omega_0$ and spread $\Delta\omega$ because of the nonuniformity of the critical currents $I_{C_j}$. Parameters used in the simulations are the (normalized) resonance frequency of the load, $\Omega = (LC)^{-1/2}/\omega_{RC}$ and the quality factor $Q = (L/R)\Omega\omega_{RC}$. The detuning parameter is defined as $\delta = \omega_0 - \omega$. We choose the frequency of the resonator to be detuned from $\omega_0$, $\Omega = \omega_0 - \delta$, so the relevant parameter is the detuning $\delta$. Finally, the parameter $\beta_L$ that scales the coupling to the resonator is set to 500 through the simulations.

We first discuss numerical simulations of equations (1), using a 4th order Runge-Kutta integration algorithm. Figure 3a shows current-voltage ($IV$) characteristics obtained by increasing and decreasing the bias current. Similarly to the experiment, the $IV$ curves are strongly hysteretic and it is possible to bias a controlled number of junctions, $N_A$, onto the resonant state. The simulation parameters are $N_T = 30$, $\beta_c = 10$, $Q = 200$, $\Omega = 1.62$. In order to compare the different steps, Figure 3b shows normalized $I-V$ characteristics. The voltage shown is the voltage drop across the array, divided by the number of junctions biased on the voltage state, $N_V$, *i.e.* it is the average voltage across each junction. (The number of junctions biased one the voltage state, $N_V$, is in general different from the number of *active* junctions, $N_A$, the junctions biased on the resonant state, because $N_V$ includes $N_A$ as well as the junctions biased on the McCumber branch.) In this representation the cavity steps appear superimposed, instead of being horizontally shifted as in the Figures 1a and 3a, where we display the total voltage across the array, but the two types of $IV$ curves are equivalent.

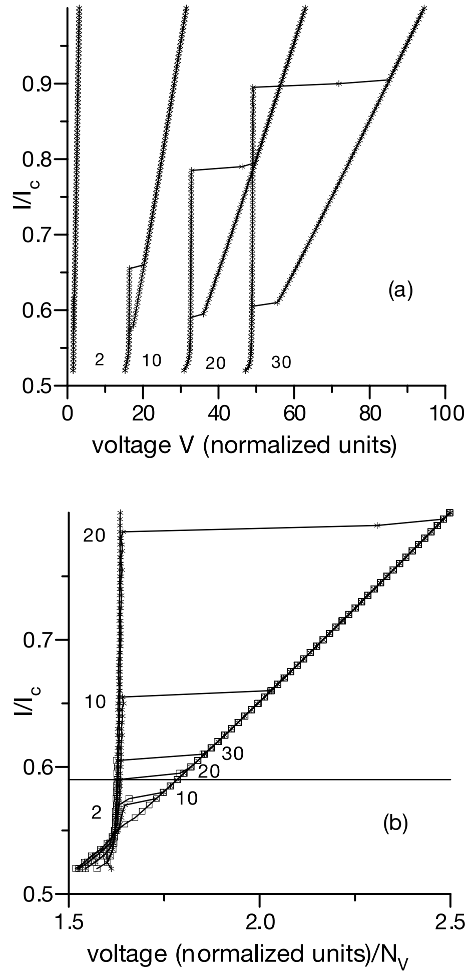


**Fig. 3.** (a) Simulated $IV$ curve of the array with increasing and decreasing bias current, as a function of the number of active junctions $N_A$ ($N_A = 2$, 10, 20, 30). The switching point from the step to the McCumber solution is shown for $N_A = 2$, 10 and 20, and indicated on the vertical bias step on the left. (b) Normalized $IV$ curve: The voltage shown is the voltage drop across the array, divided by the number of junctions biased on the voltage state. The return point from the McCumber solution to the vertical bias step is shown for $N_A = 10$, 20 and 30, and indicated on the McCumber $IV$ branch. Parameters are $N_T = 30$, $\beta_c = 10$, $\Omega = 1.62$, $\delta = 0.2$, and $Q = 200$.

Figures 3 clearly show that this one-dimensional model can produce one of the main features of the experiment: the hysteretic resonant steps. However, a closer comparison between simulation and experiment reveals some differences. The height of the steps in the calculated $IV$ curves grows dramatically when increasing the number of junctions biased on the resonant state, the active junctions. When only two junctions are active, the step is just a small feature, barely distinguishable from the McCumber branch. When more junctions are activated, the height of the step grows steadily up to a step size that is bigger by more than one order of magnitude when all the junctions are active. This behavior is not found in the experiment, where the size of the step does not show an obvious dependence on the number of active junctions. In fact, in Figure 1a, the step corresponding to just one row of junctions biased on the resonant state is roughly as large as the resonant step corresponding to ten rows active.

This difference is quite important when biasing the array and measuring the power output as a function of the number of active junctions. Experimentally, it is possible to choose a value of bias current which intersects *all* the resonant steps in the $IV$ curve, because the current range of the steps overlap in a wide interval of current values, *i.e.* it is easy to find a value of bias current common to all the resonant steps and it is possible to bias an increasing number of junctions on the resonant step by keeping the bias current fixed. Numerically, this is very difficult because the steps of lower number cover a very small current range. The current range in which the simulated steps are stable increases steadily when increasing the number of active junction, but only values of bias current belonging to the small current range of the low-number steps will be common to all the simulated steps.

In the following we describe in detail the shape of the steps in the simulated $IV$ curve and the degree of coherence of the junctions when biased on the step.

The hysteresis grows with increased number of active junctions. As we said above, by *increasing* the bias current on a step, the current where the switching from the resonant step to the McCumber curve occurs increases with the number of active junctions, that is, higher-number steps persist to higher currents. This is shown in Figure 3b. Similarly, as we *decrease* the bias current starting on the McCumber curve (where the array is oscillating incoherently), for some bias $I_{return}$ the McCumber curve becomes unstable and the array switches back to the resonant step. In other words, the collection of incoherent oscillators spontaneously switches to the resonant step, where they emit coherently with a lower frequency (the cavity resonance frequency). We note from Figure 3b that $I_{return}$ increases with the number of active junctions. More junctions will build up more (incoherent) power in the cavity that interacts back on to the junctions and makes a switch more likely – *i.e.* more oscillating junctions raise $I_{return}$. These observations show that step stability increases with increasing step number.

Figure 4 shows the stable and unstable regions of the simulations as a function of $N_V$, for the parameters of

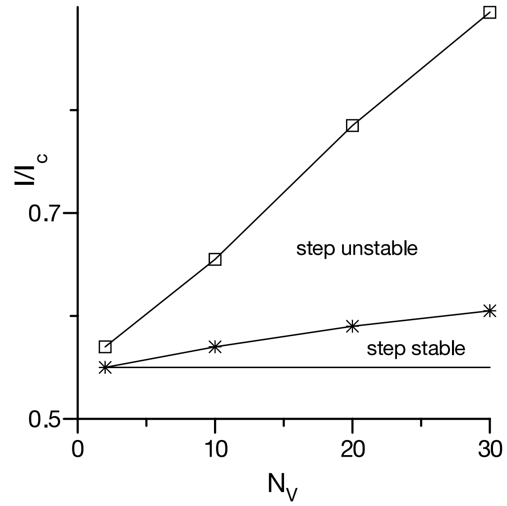


**Fig. 4.** The stable and unstable regions of the simulated $I-V$ curves. Squares denote the maximum current at which the phase-locked solution exists, if the bias is increased. Stars denote the lowest current at which the McCumber solution exists, if the bias is decreased. The dotted line denotes the bottom of the cavity induced current step. Parameters are $N_T = 30$, $\beta_c = 10$, $\Omega = 1.62$, $\delta = 0.2$, and $Q = 200$.

Figure 3a. This type of stability diagram has also been used in connection with zero field steps in long Josephson junctions, although not drawn as explicitly as in Figure 4. (See for example Ref. [14].) One clear trend is that in the simulations the range of current in which the resonant step is stable increases when increasing the number of active junctions.

To characterize the degree of coherence of the oscillating junctions, we can numerically calculate the Kuramoto order parameter [15], $r$, for the array:

$$r = \frac{1}{N_A}\sum_{j=1}^{N_A} \exp(\mathrm{i}\phi_j).$$

This order parameter $r$ is convenient because when it is zero indicates that the relative phases of the oscillators are completely disordered, and reaches 1 when the oscillators are perfectly synchronized.

We can understand the meaning of $r$ by considering Figure 1b: At the bottom of the curve, before the threshold, no AC power is detected but a DC voltage is measured. One can infer *via* the Josephson relation that the oscillators are oscillating, but the absence of measurable emitted power suggests that the phase relation is random, and therefore *via* equation (2) one can estimate that the magnitude of the order parameter is very close to 0. Above threshold, where power is detected, the magnitude of the order parameter is larger, indicating that the phases are coherent.

We next consider consequences of the stability when the array is biased at a fixed bias current. If the bias current is not too small, resonant steps will be unstable for a small number of junctions and will become stable when more junctions are biased. This can be seen in Figure 3a. If the bias current is fixed at $I/I_C = 0.59$, corresponding to the horizontal dotted line, a small number of junctions can not be biased on the resonant step, because the steps do not extend up to $I/I_C = 0.59$. The junctions are therefore biased on the McCumber branch, at voltage higher than the maximum voltage of the step. When increasing the number of junctions biased on the McCumber branch, the range of bias current in which the step is stable increases and eventually, when 11 junctions are switched, the step becomes stable for $I/I_C = 0.59$. Therefore, only when 11 junctions are in the non-zero voltage state it is possible to bias them onto the resonant step.

Because of this stability pattern, the magnitude of the order parameter $r$, as well as the power in the cavity, calculated at $I/I_C = 0.59$ as a function of the number of junctions with non-zero voltage, $N_V$, shows something resembling a threshold, as seen in Figure 5a. When fewer than 11 junctions have non-zero voltage, the power is essentially zero, because these junctions are biased on the incoherent McCumber branch; they are not on the resonant step because of the instability. When the number of active junctions is above 11, the junctions can be biased on the resonant step and they emit power coherently, *i.e.* here $N_V = N_A$, and the Kuramoto parameter in this region is equal to one.

While at first glance, there is a strong similarity between Figures 5a and 1b, this similarity is misleading. To demonstrate this, we show, in Figure 5b, another curve where the current is fixed at $I/I_C = 0.54$. Because the current is lower, the stability plot of Figure 4 allows lower-number steps to occur which do not occur at the higher current. As a consequence, power is emitted, and $|r| = 1$, as soon as two junctions are biased, $N_V 2$. We thus see that the simulated data has a kind of threshold that depends on the current through the array. This simulated threshold is $N_V = 11$ in Figure 5a, and is $N_V \geq 2$ in Figure 5b, where the bias current is lower. This simulated threshold is a consequence of the fact that simulated steps become larger (see Fig. 3) and are stable over a wider range of current (see Fig. 4).

By contrast, the experimental steps, Figure 1a, are roughly the same height. Close examination of the experimental and numerical data revealed a striking difference: In the simulations, if the system is biased on any step it is coherent, with the magnitude of the Kuramoto parameter of nearly 1 for $N_A \geq 1$. In the experiment shown in Figure 1a, when the array is biased on a low-number step, *i.e.* $14 \geq N_A \geq 1$, it is incoherent. When the experimental arrays are biased on a high-number step, they are coherent. In the experiment there is a well-defined threshold number of active junctions, not dependent upon the specific value of bias current, which separates the different types of behavior.

## 3 Summary and conclusions

We have shown that a circuit model for an array of slightly dissimilar Josephson junctions in a cavity can successfully

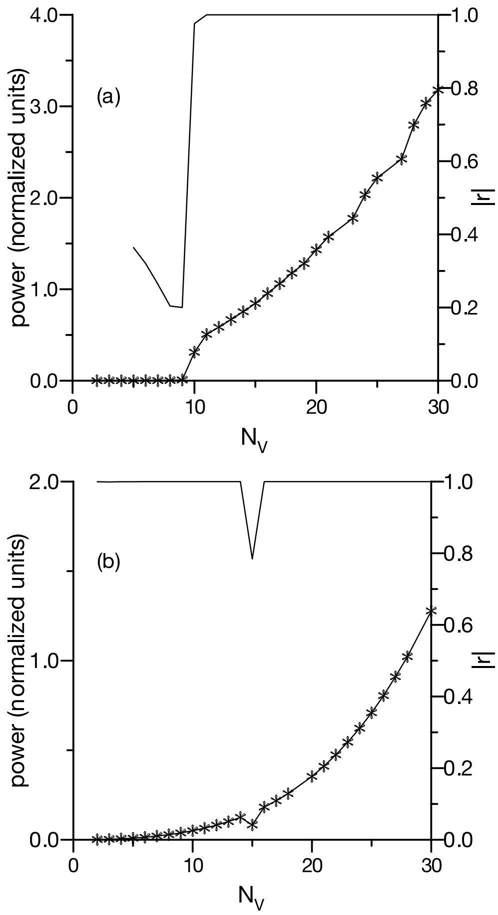


**Fig. 5.** Power coupled to the load (left axis) and the magnitude of the Kuramoto parameter (right axis) as a function of the number of active junctions, for two different values of bias current: (a) $I/I_C = 0.59$; (b) $I/I_C = 0.54$. The array parameters are the same as in Figure 3.

explain many features of the experiments in the coherent regime. Steps appear in the simulated $IV$ characteristics and the coupling of the oscillators can be quantified with a Kuramoto order parameter. The simulations also show that more power is emitted when more junctions are on the steps, or when more current flows through the junctions, again in agreement with experiment. The simulations also show that low-number steps are less stable than high-number steps.

The outstanding problem is to explain the experimental threshold. There is no threshold observed in the simulations of the type observed in experiment: In the experiments, it is possible to be biased on a step without coherent power being emitted, while in the simulations the Kuramoto order parameter is essentially one, and coherent power is emitted, on it any step. The behavior in the below-threshold region, where junctions are both biased on a resonant step and incoherent, could not be found in this model. This suggests that there is an intrinsic difference between the experiment and the model. In the model the resonant step is a result of global coupling, therefore the junctions are expected to be coherent when biased on the resonant step, as confirmed by the simulations.

Other models [12] describing frequency locking of an array to an external cavity also find that the junctions are always coherent when biased on the resonant step [16]. In the experiment, below threshold, the junctions are not coherent on the resonant step, which suggests that, below threshold, the resonant step is not generated by a global coupling mechanism. The difference between experiment and theory is not a dimensionality effect, since our two-dimensional simulations are very similar to our one-dimensional simulations. Future work should investigate the array as a transmission line with distributed coupling [7], rather than lumped, and include the groundplane.

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16. E. Almaas, D. Stroud (private communication) have informed us that the threshold reported in the simulations of reference [12] is of the same type as we find in our simulations, and is due to the fact that at fixed current it is impossible to bias on low-number steps. When they bias on any step in their simulations, the find that phase locking occurs, in agreement with our results